Review on Week 2

Defintion of a Sequence

Definition (c.f. Definition 3.1.1)**.** Formally speaking, a *sequence of real numbers*, or simply a *sequence* is a function $X : \mathbb{N} \to \mathbb{R}$ whose domain is the set of natural numbers $\mathbb{N}$ and ranges in $\mathbb{R}$. In other words, the sequence X assigns each natural number $n = 1, 2, 3, ...$ a real number. We usually denote a sequence by

$$X = (x_n) \quad \text{or} \quad X = (x_1, x_2, x_3, ...),$$

where x_n is the n-th term of the sequence.

Remark. Sequences are not sets, the **order** of the terms matter. For instance, the sequences $X = (1, 2, 3, 1, 2, 3, ...)$ and $Y = (3, 2, 1, 3, 2, 1, ...)$ are not the same although they both contain 1, 2 and 3 as their elements.

Example (c.f. Example 3.1.2)**.** The following are some examples of sequences.

- Let $b \in \mathbb{R}$. The sequence $B = (a, a, a, ...)$ is called a *constant sequence*. i.e., every term in the sequence is the same.
- The sequence $X = (1/2^n)$ represents the sequence $(1/2, 1/4, 1/8, ...)$. In this case, the sequence X is given by a **formula**.
- The *Fibonacci sequence* $F = (f_n) = (1, 1, 2, 3, 5, 8, ...)$ is given by

  $$f_1 = 1, \quad f_2 = 1, \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 3.$$

  In this case, the sequence F is given by an **inductive formula**.

Limit of a Sequence

Definition (c.f. Definition 3.1.3)**.** Let $X = (x_n)$ be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. X is said to *converge* to x if for every $\varepsilon > 0$, there exist a natural number N such that

$$|x_n - x| < \varepsilon, \qquad \forall n \geq N.$$

In this case, x is said to be the *limit* of (x_n) and denoted by $x = \lim(x_n)$.

A sequence is said to be *convergent* if it has a limit and *divergent* if it is not convergent.

Remark. Notice the following:

- In the definition, the number $x \in \mathbb{R}$ is first specified and then proven to be the limit. In other words, we have to make a "guess" of the limit of the sequence first.
- The limit of a sequence is unique (c.f. 3.1.4 Uniqueness of Limits). i.e. If x and y are both the limit of a sequence, then $x = y$.

- Sometimes we may need to show that a sequence is divergent. i.e., do not converge to any $x \in \mathbb{R}$. We have to show that for any $x \in \mathbb{R}$, there exists an $\varepsilon > 0$ such that for any natural number N, there exists some $n \geq N$ such that $|x_n - x| \geq \varepsilon$.

Example 1 (c.f. Example 3.1.6(a))**.** $\lim(1/n) = 0$.

Solution. We need to show that for every $\varepsilon > 0$, there exists a natural number N such that

$$\left|\frac{1}{n} - 0\right| < \varepsilon, \qquad \forall n \geq N.$$

Note that if $n \geq N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N}.$$

Let $\varepsilon > 0$. By **Archimedean Property**, there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Therefore

$$\left|\frac{1}{n} - 0\right| \leq \frac{1}{N} < \varepsilon, \qquad \forall n \geq N.$$

Example 2 (c.f. Example 3.1.6(d))**.** $\lim(\sqrt{n+1} - \sqrt{n}) = 0$.

Solution. We need to show that for every $\varepsilon > 0$, there exists a natural number N such that

$$\left|(\sqrt{n+1} - \sqrt{n}) - 0\right| < \varepsilon, \qquad \forall n \geq N.$$

After some calculations, we have

$$|(\sqrt{n+1} - \sqrt{n}) - 0| = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Hence if $n \geq N$,

$$|(\sqrt{n+1} - \sqrt{n}) - 0| = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}}.$$

Let $\varepsilon > 0$. By **Archimedean Property**, there exists $N \in \mathbb{N}$ such that $N > 1/\varepsilon^2$. Therefore

$$|(\sqrt{n+1} - \sqrt{n}) - 0| \leq \frac{1}{\sqrt{N}} < \varepsilon, \qquad \forall n \geq N.$$

Example 3 (c.f. Example 3.1.7)**.** The sequence $(x_n) = (0, 2, 0, 2, ...)$ is divergent.

Solution. We need to show that the sequence will not converge to any number $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ be any real number.

Now we need to choose a suitable $\varepsilon > 0$ such that whenever $N \in \mathbb{N}$, there exists $n \geq N$ such that $|x_n - x| \geq \varepsilon$.

Note that every odd term of the sequence is 0 and every even term of the sequence is 2. Hence no matter x is close to 0 or 2, we can pick a term in the sequence away from x.

Take $\varepsilon = 1$. For any natural number N, take n to be an even number greater than N if $x \leq 1$ and take n to be an odd number greater than N if $x > 1$. Then

- if $x \leq 1$, $|x_n - x| = |2 - x| = 2 - x \geq 1 = \varepsilon$.
- if $x > 1$, $|x_n - x| = |0 - x| = x \geq 1 = \varepsilon$.

Hence in any cases, there exists $n \geq N$ such that $|x_n - x| \geq \varepsilon$.

Exercises

Question 1. Determine the limit of the sequence $X = (x_n)$, in the following cases. (Such limit may not exist.)

(a) $x_n = 1 + (-1)^n$

(b) $x_n = \ln(2n)/\ln(n)$

(c) $x_n = \dfrac{1}{n(n+1)}$

(d) $x_n = \dfrac{4n-3}{2n-7}$

(e) $x_n = \sin x/n$, where $x \in \mathbb{R}$.

(f) $x_n = 2^n/n^2$

Solution. As a warm up exercise, no proofs are needed.

(a) Divergent.

(b) 1.

(c) 0.

(d) 2.

(e) ~~six.~~ 0.

(f) Divergent.

Question 2. Prove your assertion of 1(d).

Solution. Note that if $n \geq N \geq 4$, (≥ 4 to make sure the denominator positive.)

$$\left|\frac{4n-3}{2n-7} - 2\right| = \left|\frac{11}{2n-7}\right| = \frac{11}{2n-7} \leq \frac{11}{2N-7}.$$

Let $\varepsilon > 0$. By Archimedean Property, there exists a natural number N such that

$$N > \frac{1}{2}\left(\frac{11}{\varepsilon} + 7\right) \quad \text{and} \quad N \geq 4.$$

Hence if $n \geq N$,

$$\left|\frac{4n-3}{2n-7} - 2\right| \leq \frac{11}{2n-7} \leq \frac{11}{2N-7} < \varepsilon, \qquad \forall n \geq N.$$

Question 3 (c.f. Section 3.1, Ex.14)**.** Let $b \in \mathbb{R}$ satisfies $0 < b < 1$. Show that $\lim(nb^n) = 0$.

Solution. Let $a = (1/b) - 1$. Then $a > 0$ and $b = 1/(1+a)$. Hence

$$|nb^n - 0| = nb^n = \frac{n}{(1+a)^n}.$$

By Binomial Theorem, if $n \geq 2$, (≥ 2 to make sure the a^2 term exist.)

$$(1+a)^n = 1 + na + \frac{1}{2}n(n-1)a^2 + \cdots \geq \frac{1}{2}n(n-1)a^2.$$

Hence if $n \geq N \geq 2$,

$$|nb^n - 0| \leq \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2} \leq \frac{2}{(N-1)a^2}.$$

Let $\varepsilon > 0$. By **Archimedean Property**, there exists a natural number N such that

$$N > \frac{2}{a^2\varepsilon} + 1 \quad \text{and} \quad N \geq 2.$$

Hence if $n \geq N$,

$$|nb^n - 0| \leq \frac{2}{(N-1)a^2} < \varepsilon, \qquad \forall n \geq N.$$